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6. QUATERNIONIC KÄHLER MANIFOLDS

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A quaternionic Kähler manifold is a riemannian manifold of dimension $4n$ whose holonomy group is contained in $Sp_1 \cdot Sp_n$. A better description of such manifolds is the following. Let M be a riemannian $4n$-manifold $(n > 1)$. Then an almost quaternionic Kähler structure is a 3-dimensional subbundle $\mathfrak{g} \subset \mathrm{SkewEnd}(TM,TM)$ which at each point x has an orthonormal basis J_1, J_2, J_3 such that $J_j^2 = -1$, $J_j J_k = -J_k J_j$ for $j = k$, and $J_1 J_2 = J_3$. There is a natural bundle isometry

$$\mathrm{SkewEnd}(TM,TM) \xrightarrow{\cong} \Lambda^2 TM$$

which associates to each J_k a non-degenerate 2-form ω_k where

$$\omega_k(V,W) = \langle J_k V, W \rangle, \quad k = 1,2,3.$$

The 4-form

$$\Omega \equiv \omega_1^2 + \omega_2^2 + \omega_3^2$$

is independent of the choice of local frame field $\omega_1, \omega_2, \omega_3$ and is globally defined on M. The subgroup $G_x \equiv \{L \in SO(T_x M) : L^*\Omega = \Omega\}$ is isometric to $Sp_1 \cdot Sp_n = Sp_1 \times Sp_n / \mathbb{Z}_2$. The manifold M is quaternionic Kähler if and only if

$$(*) \qquad \nabla \Omega \equiv 0$$

The Riemann curvature tensor is a symmetric map $R : \Lambda^2 TM \longrightarrow \Lambda^2 TM$, and condition $(*)$ implies that

$$R\big|_{\mathfrak{g}} = \lambda \, \mathrm{Id}_{\mathfrak{g}}$$

where λ is a universal positive multiple of the scalar curvature of M. In other words, with respect to the decomposition $\Lambda^2 TM = \mathfrak{g} \oplus \mathfrak{g}^{\perp}$, we have

$$(**) \qquad R = \left(\begin{array}{c|c} \lambda & 0 \\ \hline 0 & * \end{array}\right)$$

When $\dim(M) = 4$, condition $(*)$ is trivial and $\mathfrak{G} = \Lambda^2_+$. However, we shall define a 4-manifold M to be <u>quaternionic Kahler</u> if $(**)$ holds, i.e., if M is Einstein and anti-self-dual.

The main point of this paper is to define a momentum mapping for quaternionic Kähler manifolds and to describe a process of "quaternionic reduction" which produces new quaternionic Kähler manifolds from given ones. More specifically, let M be a quaternionic Kähler manifold of dimension $4n > 4$, and suppose $S^1 \subset \mathrm{Aut}(M)$ is an S^1-subgroup generated by a Killing vector field V which satisfies $L_V \Omega = 0$. Let $\Theta_V = i_V \Omega$ be the 3-form obtained by contraction with V. With respect to a local frame field $\omega_1, \omega_2, \omega_3$ as above, we have $\Theta_V = \sum_j (i_V \omega_j) \wedge \omega_j \cong \sum_j (i_V \omega_j) \otimes \omega_j$, and so Θ_V can be considered as a 1-form with values in $\mathfrak{G}$ (an element of $\Omega^1(\mathfrak{G})$).

<u>Theorem 1</u>. <u>If</u> $\lambda \neq 0$, <u>then there exists a unique section</u> $\mu_V \in \Omega^0(\mathfrak{G})$ <u>such that</u>

$$\nabla \mu_V = \Theta_V$$

Let $\mathfrak{g} \subset \mathfrak{X}_M$ be the Lie algebra of a compact subgroup G of the automorphism group of M. Then Theorem 1 gives a <u>momentum mapping</u>

$$\mu \in \Omega^0(\mathfrak{g}^* \otimes \mathfrak{G})$$

with the equivariance property

$$g_*(\mu_V(x)) = \mu_{\mathrm{Ad}_g(V)}(gx)$$

(provided that $\lambda \neq 0$). The automatic nature of the existence and equivariance of μ is strikingly different from the "abelian" case where $\lambda = 0$. Let $Z_G \equiv \{x \in M : \mu(x) = 0\}$.

<u>Theorem 2</u>. <u>At all regular points, the manifold</u> Z_G/G <u>with its naturally induced metric, is quaternionic Kähler</u>.

<u>Corollary 3</u>. <u>Let</u> $G \cong S^1 \subset \mathrm{Aut}(M)$ <u>be generated by the vector field</u> V <u>and suppose that</u> $V \neq 0$ <u>along</u> Z_G. <u>Then</u> Z_G/G <u>is a quaternionic Kähler orbifold of dimension</u> $4n-4$.

Starting with $\mathbb{P}^n_{\mathbb{H}}$ we obtain a large number of examples. In particular, we prove the following. For relatively prime integers a,b,c let $\mathbb{P}^2_{a,b,c}$ be the <u>weighted projective plane</u> defined as $(\mathbb{C}^3-\{0\})/\mathbb{C}^\times$ where $\mathbb{C}^\times$ acts by $\varphi_t(x,y,z) = (t^a x, t^b y, t^c z)$. Each $\mathbb{P}^2_{a,b,c}$ is a compact simply-connected orbifold.

<u>Theorem 4</u>. <u>Let</u> $p,q \in \mathbb{Z}^+$ <u>be relatively prime integers with</u> $q < p$. <u>Then each of the weighted projective planes</u>

$$\mathbb{P}^2_{p+q,p+q,2p} \qquad \underline{\text{if}}\ p+q\ \underline{\text{is odd}}$$

$$\mathbb{P}^2_{\frac{p+q}{2},\frac{p+q}{2},p} \qquad \underline{\text{if}}\ p+q\ \underline{\text{is even}}$$

<u>carries a (non-locally symmetric) riemannian orbifold metric which is Einstein, anti-self-dual, and of positive scalar curvature</u>.

<u>Note.</u> N. Hitchin has proved that any simply-connected riemannian 4-manifold which is quaternionic Kähler with $\lambda > 0$ is S^4 or $\mathbb{P}^2_{\mathbb{C}}$, with the standard symmetric metrics. Hence, the appearance of singularities in dimension 4 is necessary.

Similar examples in all dimensions $4n$ are constructed.

This represents work done in collaboration with K. Galicki at I.T.P. in Stony Brook who pioneered the construction. The first suggestion that such a reduction process might exist came from M. Roček, also at I.T.P..